2.7d Liapunov exponents

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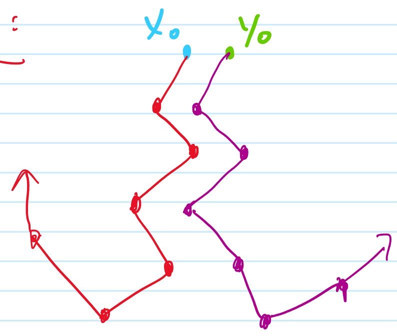
Def. 2.7 Let $f: I \to I$. Then the map f has **sensitive dependence** on the initial condition x_0 if $\exists\, \delta > 0$ s.t. $\forall\, \varepsilon > 0$, $\exists\, y_0 \in I$ and $k \in \mathbb{N}$ s.t.

$$|x_0 - y_0| < \varepsilon \quad \text{and} \quad |f^k(x_0) - f^k(y_0)| > \delta.$$

Illustration:



Trajectories that start near x_0 don't stay near each other

For a given initial condition x_0,

let $\Delta(\varepsilon, t) = \left| f^t(x_0 + \varepsilon) - f^t(x_0) \right|$

Suppose $x_{t+1} = 2x_t$ as a simple example, with $x_0 = 0$.

Then $\Delta(\varepsilon, t) = \left| 2^t(x_0 + \varepsilon) - 2^t x_0 \right| = \left| 2^t \varepsilon \right| = \left| \varepsilon e^{t \ln 2} \right|$

how fast the 2 trajectories diverge

- $\Delta(\varepsilon, t)$ should have linear dependence on ε
- $\Delta(\varepsilon, t)$ should have exponential dependence on t.

Thus, we can consider defining the **Liapunov exponent** $\lambda(x_0)$ by

$$\varepsilon e^{t\lambda(x_0)} \approx \left| f^t(x_0 + \varepsilon) - f^t(x_0) \right|,$$ where ε is small and t is large

Let $\varepsilon \to 0$

$$e^{t\lambda(x_0)} \approx \lim_{\varepsilon \to 0} \left| \frac{f^t(x_0 + \varepsilon) - f^t(x_0)}{\varepsilon} \right|$$

$$= \left| \frac{d}{dx} f^t(x_0) \right|$$

$$\Rightarrow \quad \lambda(x_0) = \frac{1}{t} \ln \left| \frac{d}{dx} f^t(x_0) \right|$$

$$= \frac{1}{t} \ln \left| f'(x_0) f'(x_1) \cdots f'(x_{t-1}) \right|$$

$$= \frac{1}{t} \sum_{k=0}^{t-1} \ln |f'(x_k)|.$$

But still have t-dependence, so let $t \to \infty$

$$\lambda(x_0) = \lim_{t \to \infty} \frac{1}{t} \sum_{k=0}^{t-1} \ln |f'(x_k)|.$$

Def 2.8 The **Liapunov exponent** (**Lyapunov exponent**) at x_0 of $x_{t+1} = f(x_t)$ is

$$\lambda(x_0) = \lim_{t \to \infty} \frac{1}{t} \sum_{k=0}^{t-1} \ln |f'(x_k)|$$

If $\lambda(x_0)$ is independent of x_0, then $\lambda(x_0)$ is denoted by λ and referred to as the Lyapunov exp of f. There exist sensitive dependence on initial conditions if $\lambda(x_0) > 0$ for all initial conditions in the domain.

Aside: Can verify all locally asymptotically stable equilibria have negative Lyapunov exponents
unstable equilibria have positive Lyapunov exponents.

Ex. Let $x_{t+1} = \frac{x_t + x_t^2}{2}$. $\bar{x} = 0, 1$ are the equilibria.

Let's compute Liapunov exponents at the equilibria.

$$f(x) = \frac{x + x^2}{2}, \qquad f'(x) = \frac{1}{2} + x$$

$$f'(0) = \frac{1}{2} \qquad \text{(stable)}$$

$$f'(1) = \frac{3}{2} \qquad \text{(unstable)}$$

Then $\lambda(0) = \lim_{t \to \infty} \frac{1}{t} \sum_{k=0}^{t-1} \ln |f'(0)| = \ln \frac{1}{2} = -\ln 2 < 0$

$$\lambda(1) = \lim_{t\to\infty} \frac{1}{t}\sum_{k=0}^{t-1} \ln|f'(1)| = \ln\frac{3}{2} = \ln 3 - \ln 2 > 0.$$

In practice: The Liapunov exponents for nonequilibria are harder to compute, so we often approximate them. Note, for fixed j, $\underbrace{\lambda(x_0)}=\underbrace{\lambda(x_j)}$,

$$\lambda(x_0) \approx \frac{1}{N}\sum_{k=j}^{k=j+N-1} \ln|f'(x_k)|$$

$\lim_{t\to\infty} \frac{1}{t}\sum_{k=0}^{t-1} \ln|f'(k)|$

Aside: We can generalize to system of first-order equations

$$X(t+1) = F(X(t))$$

by using the Jacobian matrix $J(X(t))$, and looking at the spectral radius ρ.

Lyapunov exp $\lambda(X(0)) = \lim_{t\to\infty} \ln \rho\left(\prod_{k=0}^{t-1} J(X(k))\right).$

If $\lambda(X(0)) > 0$, then the solution to the system exhibits sensitive dependence on the initial condition $X(0)$.

If the solution is not asymptotically periodic, and no Lyapunov exp is exactly 0, then the solution is chaotic.